Section 7.2: Equally Likely outcomes

Calculation of probabilities in real life is very difficult. One helpful method of calculation, is to create a sample space with equally likely outcomes and assign probabilities to the outcomes based on this assumption. Examples of experiments with equally likely outcomes are:

- Draw a random sample of size n from a population or urn. The assumption that the sample is drawn at random means that all samples of size n have an equal chance of being chosen (much of statistical analysis depends on the assumption that samples are chosen randomly).
- Flip a fair coin n times and observe the sequence of heads an tails that results.
- Roll n dice, die 1, die 2, die 3, . . . , die n, and observe the ordered sequence of numbers on the uppermost faces.

Equally Likely Outcomes

For any sample space with N equally likely outcomes, we assign the probability $\frac{1}{N}$ to each outcome.

Example Experiment: Flip a fair coin. The sample space for this experiment has two equally likely outcomes: $S = \{H, T\}$. Assign probabilities to these outcomes.

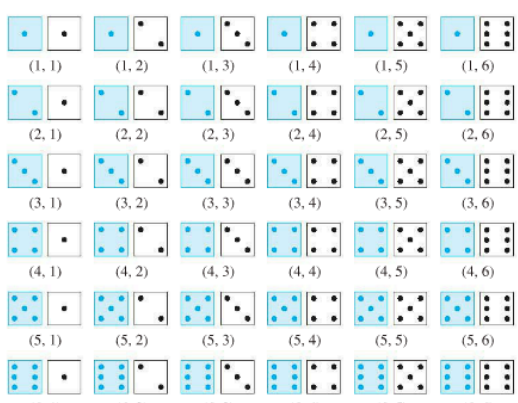
Example Experiment: Flip a fair coin twice and record the sequence of Heads and tails. Each of the four outcomes: {HH, HT, TH, TT } have the same probability.

(a) What is the probability that TT is the outcome of this experiment?

If E is an **event** in a sample space, S, with N **equally likely (simple) outcomes,** the probability that E will occur is the sum of the probabilities of the outcomes in E, which gives

$$\boxed{P(E) = \frac{\text{the number of outcomes in } E}{\text{the number of outcomes in S}}} = \frac{n(E)}{n(S)} = \frac{n(E)}{N}$$

Example A pair of fair six sided dice, one blue and one white, are rolled and the pair of numbers on the uppermost face is observed. We record blue first and then white. The sample space for the experiment is shown below:



$$\text{Sample Space: } = \begin{array}{cccccc} \{(1,1) & (1,2) & (1,3) & (1,4) & (1,5) & (1,6) \\ (2,1) & (2,2) & (2,3) & (2,4) & (2,5) & (2,6) \\ (3,1) & (3,2) & (3,3) & (3,4) & (3,5) & (3,6) \\ (4,1) & (4,2) & (4,3) & (4,4) & (4,5) & (4,6) \\ (5,1) & (5,2) & (5,3) & (5,4) & (5,5) & (5,6) \\ (6,1) & (6,2) & (6,3) & (6,4) & (6,5) & (6,6)\} \end{array}$$

(a) Let E be the event that the numbers observed add to 7.

$$\begin{array}{c} \text{E} \\ \text{in red} \end{array} = \begin{array}{cccccc} \{(1,1) & (1,2) & (1,3) & (1,4) & (1,5) & \color{red}{(1,6)} \\ (2,1) & (2,2) & (2,3) & (2,4) & \color{red}{(2,5)} & (2,6) \\ (3,1) & (3,2) & (3,3) & \color{red}{(3,4)} & (3,5) & (3,6) \\ (4,1) & (4,2) & \color{red}{(4,3)} & (4,4) & (4,5) & (4,6) \\ (5,1) & \color{red}{(5,2)} & (5,3) & (5,4) & (5,5) & (5,6) \\ \color{red}{(6,1)} & (6,2) & (6,3) & (6,4) & (6,5) & (6,6)\} \end{array}$$

What is the probability of E?

(b) Let F be the event that the numbers observed add to 11. List the elements of the set F and calculate $P(E)$.
`Simulation`

(c) Let G be the event that the numbers on both dice are the same. What is $P(G)$?

Example A pair of fair dice, one six sided and one four sided are rolled and the pair of numbers on the uppermost face is observed. The sample space shown below has equally likely outcomes. Calculate the probability of the event "The numbers observed add to 4".

$$\text{Sample Space: } = \begin{array}{cccc} \{(1,1) & (1,2) & (1,3) & (1,4) \\ (2,1) & (2,2) & (2,3) & (2,4) \\ (3,1) & (3,2) & (3,3) & (3,4) \\ (4,1) & (4,2) & (4,3) & (4,4) \\ (5,1) & (5,2) & (5,3) & (5,4) \\ (6,1) & (6,2) & (6,3) & (6,4)\} \end{array}$$

Random Selection or **Random Outcomes** When we say that outcomes are selected randomly, it implies(by definition) that individual outcomes in the sample space are equally likely. For example, if we say we drew a random sample of size n from a population, we are assuming that our selection process gave all samples of size n an equal chance of being drawn.

Last time we saw that our intuition served us well when calculating probabilities if we draw a random sample of size one from an urn or population, where we know the contents of the urn or the composition of the population. However when we increase the sample size to two, our intuition is no longer useful and the time we invested in learning to count pays off. The formula given above translates to:

If I take a sample of size K from an urn with N objects, the probability that the sample is a sample of Type X is

$$\frac{\text{The number of samples of Type X}}{\text{The total number of samples}} = \frac{\text{The number of samples of Type X}}{C(N,K)}$$

Example Suppose I have an urn containing twelve numbered marbles, 8 of which are red and 4 of which are white. If I take a sample of two marbles (observing number and color) from the urn,

(a) What is the probability of getting two red marbles?

The "total number of samples" is $C(12,2) = \frac{12 \cdot 11}{2} = 66$.

The "number of samples with 2 red marbles" is $C(8,2) = \frac{8 \cdot 7}{2} = 28$.

The probability that a sample of two marbles are both red is $\frac{28}{66} = 0.4242\cdots$.

(b) What the probability of getting one red and one white marble in the sample?

The "total number of samples" is still $C(12,2) = 66$.
The "number of samples with 1 red marble and 1 white marble" is $C(8,1) \cdot C(4,1) = 32$.
The probability that a sample of two marbles is 1 red and 1 white is $\frac{32}{66} = 0.4848\cdots$.

Example; The Hoosier Lottery When you buy a Powerball ticket, you select 5 different white numbers from among the numbers 1 through 59 (order of selection does not matter), and one red number from among the numbers 1 through 35. What is the probability that your selection will be the winning one?

The "total number of samples" is $C(59,5) \cdot C(35,1) = 175,223,510$.
The "number of samples of Type X" in this case is 1.
Your probability of winning the lottery is $\frac{1}{175,223,510} \approx 5.70699673805187 \cdot 10^{-9}$.

Example If a `poker hand` is dealt fairly(randomly), what is the probability that a hand with three cards from one denomination and two from another (a house) will be dealt?

The "total number of samples" is $C(52, 5) = 2,598,960$.
One way to count the "number of samples of Type X" is to first pick the two denominations. Since one of them will have 3 cards and the other 2 the two denominations can be distinguished so you can do this $P(13, 2)$ ways. Then you can pick 3 cards from the first denomination in $C(4, 3)$ ways and 2 cards from the second denomination in $C(4, 2)$ ways.
Hence "number of samples of Type X" is $P(13, 2) \cdot C(4, 3) \cdot C(4, 2) = 3,744$.
Hence the probability that a hand with three cards from one denomination and two from another is dealt is $\dfrac{3,744}{2,598,960} = 0.0014405762$.

You can do a little research on the probabilities of all types of poker hands here

Example *(A respectable example) A box ready, for shipment, contains 100 light bulbs, 10 of which are defective. The quality control test is to take a random sample of 5 light bulbs, without replacement, from the box. If one is defective, the box will not be shipped. What is the probability that the box will be shipped.

The "total number of samples" is $C(100, 5) = 75,287,520$.
The "number of samples of Type X" is $C(10, 5) = 252$.
The probability that the box will be shipped is $\dfrac{252}{75287520} = 3.3471682956219 \cdot 10^{-6}$.

Example A coin is flipped 4 times and the sequence of heads and tails is recorded. All of these sequences are equally likely.

(a) How many elements are there in this sample space?

$2^4 = 16.$

(b) How many outcomes with exactly 3 heads?

$C(4,3) = C(4,1) = 4.$

(c) Let E be the event "we get exactly 3 heads", what is $Pr(E)$?

$PR(E) = \frac{4}{16} = 0.25.$

The Complement rule for sample spaces with equally likely outcomes

If E is an event in a sample space S, then $n(E) + n(E') = n(S)$, therefore

$$\frac{n(E)}{n(S)} + \frac{n(E')}{n(S)} = \frac{n(S)}{n(S)} = 1.$$

Thus we have **the complement rule:**

$$P(E) + P(E') = 1 \quad \text{or} \quad \boxed{P(E') = 1 - P(E)}$$

Note: If we define "success" to be the event that we get an outcome in E and "failure to be the event that we do not get an outcome in E, then $P(\text{success}) = 1 - P(\text{"failure"})$. (We will use this terminology later when studying the Binomial distribution.)

Example 8 Flip a coin 10 times and observe the sequence of heads and tails.

(a) How many outcomes are in this sample space?

$2^{10} = 1,024.$

(b) What is the probability that you observe 5 heads?

$C(10,5) = 30,240$

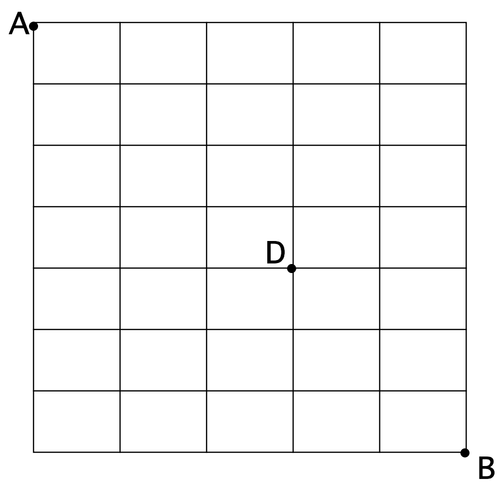
(c) What is the probability that you will observe at least one tail?

$1 -$ the probability that you will observe no tails.
The number of outcomes with no tails is $C(10,0) = 1$.
The probability that you will observe at least one tail is
$1 - \frac{1}{1024} = 0.9990234375.$

(d) What is the probability that we will observe at least two heads?

$1-$the probability that you will observe 0 or 1 heads.
The number of outcomes with no heads is
1.
The number of outcomes with one head is
$C(10,1) = 10$.
The probability that we will observe at least
two heads is $\frac{10+1}{1024} = 0.9892578125$.

Example 8 (a) Kristina, on her morning run, wants to get from point A to point B. How many routes with no backtracking can she take (she always travels South or East)?



During her run she needs to go 5 blocks east and 7 blocks south. Hence the number of routes is
$C(7+5,7) = C(12,7) = C(12,5) = 792$.

(b) If Kristina chooses a route from among those with no backtracking at random, what is the probability that she will not run past the doberman at D?

It is easier to count the routes which do go by the doberman: $C(3+4,3) \cdot C(2+3,2) = 35 \cdot 10 = 350$. Hence the probability that she will not run past the doberman is
$1 - \frac{350}{792} = 0.5580808081$.

Example * A box ready, for shipment, contains 100 light bulbs, 10 of which are defective. The quality control test is to take a random sample of 5 light bulbs, without replacement, from the box. If one is defective, the box will not be shipped. What is the probability that the box will not be shipped. (Use your answer to Example * above).

$1 - 3.3471682956219 \cdot 10^{-6} = 0.9999966528$.

Extras

Example Harry Potter's closet contains 12 numbered brooms, of which 8 are Comet Two Sixty's(numbered 1-8) and 4 are Nimbus Two Thousand's(numbered 9 -12). Harry, Ron, George and Fred want to sneak out in the middle of the night for a game of Quidditch. They are afraid to turn on the light in case Filch catches them. Harry reaches into the closet and pull out a random sample 4 brooms.

(a) Calculate the probability that all of the brooms will be Comet Two Sixty's.

From Topic 6 the number of samples is $C(12,4)$ and the number of samples in which all the brooms are Comet Two Sixty's is $C(8,4)$. Hence the answer is $\frac{C8,4)}{C(12,4)} = 0.1414141414\cdots$.

(b) What is the probability that Harry chooses a sample with exactly 4 Nimbus Two Thousand's?

$$\frac{C(4,4)}{C(12,4)} = 0.002020202$$

(c) What is the probability that Harry will have at least one Nimbus Two Thousand in his sample?

$$1 - \text{probability of 0 Nimbus Two Thousands} = 1 - \frac{1}{C(12,4)} = 0.997979798.$$

Old Exam Questions

1 An experiment consists of drawing 3 balls at random from an urn containing 2 red balls and 4 white balls. What is the probability of getting at least 2 white balls?

$$\frac{C(3,2) + C(3,3)}{C(6,3)} = \frac{1}{5}.$$

(a) $\frac{4}{5}$ (b) $\frac{2}{3}$ (c) $\frac{1}{5}$ (d) $\frac{2}{5}$ ($\underline{\text{e}}$) $\frac{3}{5}$

2 Three out of 25 new cars are selected at random to check for steering defects. Suppose that 7 of the 25 cars have such defects. What is the probability that all 3 of the selected cars are defective?

$$\frac{C(7,3)}{C(25,3)}.$$

(a) $\frac{\binom{7}{3}}{\binom{25}{3}}$ (b) $\frac{\binom{25}{3}}{\binom{25}{7}}$ (c) $\frac{\binom{18}{3}}{\binom{25}{3}}$ (d) $\frac{\binom{18}{7}}{\binom{25}{7}}$ (e) $\frac{3!}{\binom{25}{3}}$.

3 A fair coin is tossed 10 times. What is the probability of observing exactly 3 heads.

(a) $\frac{3}{10}$ (b) $\frac{P(10{,}3)}{2^{10}}$ (c) $\frac{2^3}{2^{10}}$ (d) $\frac{1}{10^3}$ (e) $\frac{C(10{,}3)}{2^{10}}$.

$$\frac{C(10,3)}{2^{10}}$$

Coincidence: The Birthday Problem

Many people are fascinated by coincidence, but often underestimate the probability of such coincidences. This is the basis of many magic tricks. One such example is the probability that two people in a group will have the same birthday (Month and Day). The following simulation allows you to assign birthdays randomly to groups of selected sizes up to 100 : Birthday Applet

The Table below shows the probability that at least two people in a group will share a birthday, according to group size under the assumption that birthdays are randomly distributed throughout the year:

Group Size	Probability at least 2 Birthdays are the same
20	0.4114
30	0.7063
40	0.8912
50	0.9704
60	0.9941
70	0.9991
80	0.9999

We will work through the calculation for a group of twenty people. We will use the complement rule:

$$Pr(E) = 1 - Pr(E').$$

Let E be the event that at least two people in the group share a birthday. Then E' is the event that the birthdays of the people in the group are all different. To calculate the probability of E', we consider the experiment of making a random list of 20 birthdays. The event E' then corresponds to the set of lists where all birthdays on the list are different.

The total number of lists of 20 birthdays we can make is 365^{20}, which is a 52 digit number:

1761397149289626339965899029490128250217437744140625

The number of lists with 20 different birthdays can be found using the multiplication principle. It is

$$365 \times 364 \times 363 \times 362 \times \cdots \times 345 = P(365, 20) =$$

1036690753342460944844102991873410166195245875200000

Now

$$Pr(E') = P(365, 20)/365^{20} = 0.5885616164$$

and hence

$$Pr(E) = 1 - Pr(E') = 0.4114383836.$$

The calculations for groups of different sizes are similar.

In this article, we see the author says that since each of the 32 teams in the 2014 World Cup had a squad of 23 players, we would expect about 16 teams to have at least two squad members with a shared birthday. This was in fact the case: The birthday paradox at the World Cup

Try the following problem, which is similar, using your calculator: (First make a guess as to what you think the probability is)

Problem If 10 people each choose a number (secretly) between 1 and 50 (inclusive), what are the chances that at least two of the numbers will be the same if everyone chooses randomly.

$$1 - \left(\frac{P(50,10)}{50^{10}}\right) \approx 0.6182933194.$$

More on coincidence

The following series of coincidences between the life events of Abraham Lincoln and John F. Kennedy often strike readers as unusual or even spooky:

Abraham Lincoln was elected to Congress in 1846.
John F. Kennedy was elected to congress in 1946.

Abraham Lincoln was elected president in 1860.
John F. Kennedy was elected president in 1960.

Lincoln's secretary was named Kennedy.
Kennedy's secretary was named Lincoln.

Andrew Johnson, who succeeded Lincoln, was born in 1808.
Lyndon Johnson, who succeeded Kennedy, was born in 1908.

John Wilkes Booth, who assassinated Lincoln, was born in 1839.
Lee Harvey Oswald, who assassinated Kennedy, was born in 1939.

Considering the amount of information available about these two men, is it truly surprising that we might find 5 coincidences if we were really looking to find such coincidences (see the example below)?

Example Suppose I create 2 fictitious characters, Mr. A and Mr. B and I give each a profile by listing 1000 life events

	Mr. A	Mr B
Born:	1920	1932
Joined Army:		
Went to Law school:		
Married:		
First Child born:		
Youngest child born:		
Secretary born:		
Dog died:		
Became president:		
returned to office:		
bought first lottery ticket:		
relative died:		
Visited Ireland:		
etc..........		

For each life event, I assign a year at random from 1900 to 1999 for that event for Mr. A and a random year from 1900 to 1999 for that event for Mr. B (same as rolling a pair of fair 100 sided dice each time we assign the pair of years to an event). Obviously the choice would be from a smaller number of years if we take the year of birth into account. This would reduce the probabilities, if anything.

(a) What is the probability that I assign the same year for both characters to any given event? (= The probability that the number on both 100 sided dice is the same).

$1 - \frac{100 \cdot 99}{100 \cdot 100} = 0.01$. Hence the probability that they are different is 0.99.

(b) How many of these 1000 life events would you expect to show the same year for both Mr. A and Mr. B.?

$\approx 1000 \cdot 0.01 = 10$

The following video discusses some common misconceptions about coincidences:
`It could just be coincidence`